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Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators

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Abstract. Spectral properties of singular Sturm-Liouville operators of the form $A = \operatorname{sgn}(\cdot)(-\frac{d^2}{dx^2} + V)$ with the indefinite weight $x \mapsto \operatorname{sgn}(x)$ on $\mathbb{R}$ are studied. For a class of potentials with $\lim_{|x|\to\infty} V(x) = 0$ the accumulation of complex and real eigenvalues of A to zero is investigated and explicit eigenvalue problems are solved numerically.

1. Introduction and main results

Let V be a locally integrable real valued function on $\mathbb{R}$, $V \in L^1_{\text{loc}}(\mathbb{R})$. Recall that the *maximal operator* T associated to the differential expression $-\frac{d^2}{dx^2} + V$ in the Hilbert space $L^2(\mathbb{R})$ is given by

$$(Tf)(x) = -f''(x) + (Vf)(x), \qquad x \in \mathbb{R}, \tag{1}$$

defined on the dense subspace

$$\operatorname{dom} T = \left\{ f \in L^2(\mathbb{R}) : f, f' \,\text{absolutely continuous}, -f'' + Vf \in L^2(\mathbb{R}) \right\}. \tag{2}$$

We note that due to Weyl's alternative (see, e.g., [21, 22]) the maximal operator T is not necessarily selfadjoint in the Hilbert space $L^2(\mathbb{R})$, but always contains selfadjoint restrictions. However, if the differential expression $-\frac{d^2}{dx^2} + V$ is in the limit point case at both singular endpoints $+\infty$ and $-\infty$, then T is already selfadjoint in $L^2(\mathbb{R})$, cf. [21, 22]. A sufficient condition on the potential V for $-\frac{d^2}{dx^2} + V$ to be in the limit point case is, e.g.,

$$\liminf_{x \to \pm\infty} \frac{1}{x^2} V(x) > -\infty, \tag{3}$$

see [21, 22]. The spectral properties of onedimensional Schrödinger operators of the form (1)-(2) play a fundamental role in quantum mechanics and have been studied by mathematicians and physicists for many decades.

Nowadays there is a strong interest in spectral analysis of non-selfadjoint second order differential operators which still satisfy certain (Krein space) symmetry properties, e.g., $\mathcal{PT}$-symmetric operators (see the comprehensive review paper [7] for an overview and further references and, e.g., [8, 19]). Of particular interest to us is the above differential expression $-\frac{d^2}{dx^2} + V$ multiplied with the indefinite weight function $\operatorname{sgn}(\cdot)$, that is, we consider the differential operator

$$(Af)(x) := \operatorname{sgn}(x)(-f''(x) + (Vf)(x)), \qquad x \in \mathbb{R}, \tag{4}$$

defined on the maximal domain $\operatorname{dom} A = \operatorname{dom} T$. Indefinite Sturm-Liouville operators of similar structure were considered in, e.g., [3, 9, 10, 14, 15]. Obviously, A is not a symmetric or selfadjoint operator in the space $L^2(\mathbb{R})$ equipped with the usual Hilbert scalar product $(\cdot, \cdot)$. For the following investigations it is convenient to introduce an indefinite inner product $[\cdot, \cdot]$ on $L^2(\mathbb{R})$ by

$$[f, g] := \int_{\mathbb{R}} f(x)\overline{g(x)} \operatorname{sgn}(x)\, dx, \qquad f, g \in L^2(\mathbb{R}). \tag{5}$$

Then we have $[f, g] = (\operatorname{sgn}(\cdot)f, g)$ and $(L^2(\mathbb{R}), [\cdot, \cdot])$ is a so-called *Krein space*, cf. [1]. The differential operator A is a densely defined closed operator in this Krein space. It is not hard to see that the operator A is selfadjoint with respect to the Krein space inner product $[\cdot, \cdot]$ if and only if the operator T is selfadjoint with respect to the usual $L^2(\mathbb{R})$ scalar product $(\cdot, \cdot)$. Recall that besides symmetry with respect to the real line the spectrum of a selfadjoint operator in a Krein space can be quite arbitrary. In particular, the whole complex plane may consist of eigenvalues or spectral points from the continuous spectrum.

The present note has the following main objectives. In the case that the indefinite Sturm-Liouville operator A in (4) is selfadjoint in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$ and the potential V tends to zero for $|x| \to \infty$, firstly the behaviour of complex and real eigenvalues of A in a neighborhood of zero is studied and secondly numerical methods are applied to analyze some explicitly solvable problems; among them $V(x) = -(1+|x|)^{-1}$.

1.1. Accumulation of eigenvalues for a class of indefinite Sturm-Liouville operators

Our first theorem, which also follows from the considerations in [3] and [15], summarizes the spectral properties of the indefinite Sturm-Liouville operator A. Recall that for a selfadjoint operator in a Krein space the *essential spectrum* σ_{ess} consists of all spectral points which are no isolated eigenvalues with finite-dimensional algebraic eigenspaces.

Theorem 1 *Suppose that* $\lim_{x\to\pm\infty} V(x) = 0$ *holds. Then the operator* A *in* (4) *is selfadjoint in the Krein space* $(L^2(\mathbb{R}), [\cdot,\cdot])$*, the essential spectrum of* A *covers the real line,* $\sigma_{\text{ess}}(A) = \mathbb{R}$*, the nonreal spectrum of* A *consists of eigenvalues and for every* $\delta > 0$ *there are at most finitely many nonreal eigenvalues of* A *in* $\{z \in \overline{\mathbb{C}} : |z| > \delta\}$*. In particular, no point of* $\overline{\mathbb{R}}\backslash\{0\}$ *is an accumulation point of nonreal eigenvalues of* A.

In the following we are interested in the behaviour of the eigenvalues of A in a neighborhood of zero. The next theorem, which is the main result of this note, shows the different possibilities for eigenvalue accumulation at zero.

Theorem 2 *Suppose that* $\lim_{x\to\pm\infty} V(x) = 0$ *holds and that* V *satisfies the condition*

$$\limsup_{x\to+\infty} x^2\, V(x) < -\frac{1}{4} \quad \text{or} \quad \limsup_{x\to-\infty} x^2\, V(x) < -\frac{1}{4}. \tag{6}$$

Then at least one of the following statements is true.

(i) *The nonreal eigenvalues of* A *accumulate to zero.*

(ii) *The growth of* $\lambda \mapsto (A-\lambda)^{-1}$ *near zero is not of finite order, i.e., for every open neighborhood* $\mathcal{O} \subset \mathbb{C}$ *and all* $m \geq 1$, $M > 1$ *there exists* $\lambda \in \rho(A) \cap \mathcal{O}\backslash\mathbb{R}$ *such that*

$$\|(A-\lambda)^{-1}\|\, |\mathrm{Im}\,\lambda|^m > M(1+|\lambda|)^{2m-2}.$$

(iii) *There exists a sequence* $(\mu_n)_{n\in\mathbb{N}} \subset (0,\infty)$ *of (embedded) eigenvalues of* A *and associated eigenvectors* $(g_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} \mu_n = 0$ *and* $[g_n, g_n] \leq 0$ *holds.*

(iv) *There exists a sequence* $(\nu_n)_{n\in\mathbb{N}} \subset (-\infty, 0)$ *of (embedded) eigenvalues of* A *and associated eigenvectors* $(h_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} \nu_n = 0$ *and* $[h_n, h_n] \geq 0$ *holds.*

The proofs of Theorem 1 and Theorem 2 make use of spectral and perturbation theory of selfadjoint operators in Krein spaces. In particular, the theory of locally definitizable operators and a result on finite rank perturbations in resolvent sense of such operators will be applied, see Section 2 for the details.

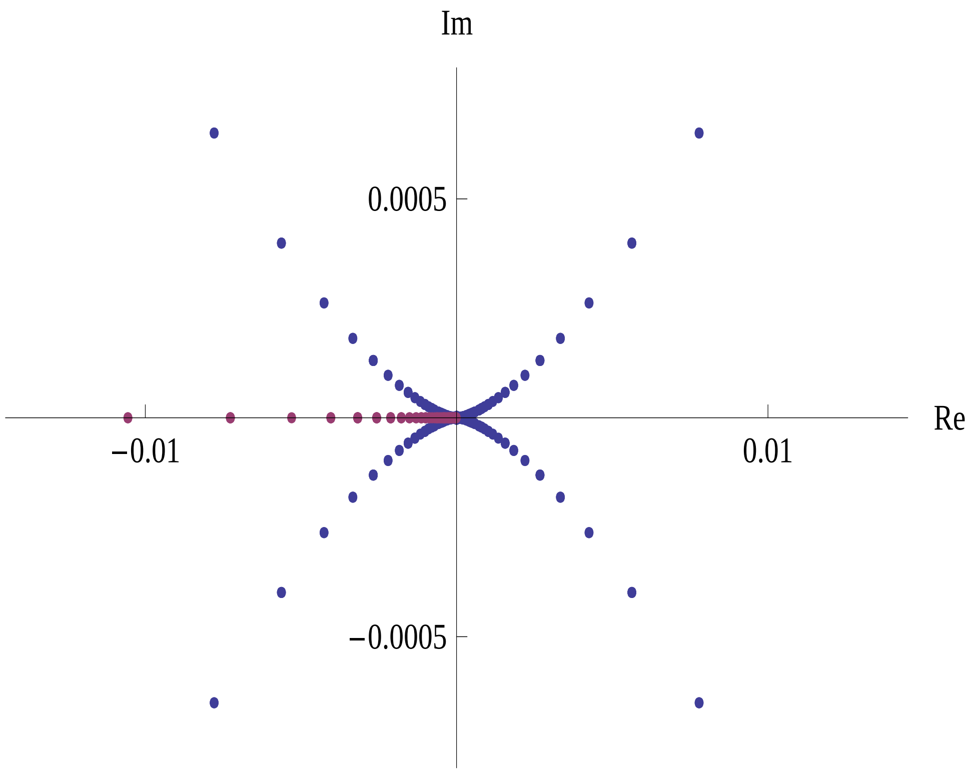

Figure 1. Accumulation of complex eigenvalues to zero of the indefinite differential operator A (blue points) and negative eigenvalues of T (red points) for the potential $V(x) = -(1+|x|)^{-1}$.

1.2. Numerical examples

In this subsection some explicit examples for potentials V are given where nonreal eigenvalues of the indefinite Sturm-Liouville operator (4) accumulate to zero. We mention that an existence result on such potentials was already proved in [15]. The models considered here arise from completely solvable models on the half axes, so that eigenvalues and eigenfunctions can be computed by using standard software packages, e.g. Mathematica (WolframResearch).

As a first example we consider

$$V(x) = -\frac{1}{1+|x|}, \qquad x \in \mathbb{R}. \tag{7}$$

Clearly, both assumptions on V from Theorem 2 are satisfied. The differential operator

$$(Tf)(x) = -f''(x) - \frac{1}{1+|x|} f(x), \qquad x \in \mathbb{R},$$

is selfadjoint in the Hilbert space $L^2(\mathbb{R})$. Obviously the operator T is semibounded from below by -1 and one verifies numerically that -0.429911 is the lower bound. Since $\sigma(T) \cap (-\infty, 0)$ consists of simple eigenvalues the point -0.429911 is an eigenvalue. Furthermore, the negative eigenvalues of T accumulate to zero (red points in Figure 1) and the half axis $[0, \infty)$ is the essential spectrum of T.

The indefinite Sturm-Liouville operator

$$(Af)(x) = \operatorname{sgn}(x)\left(-f''(x) - \frac{1}{1+|x|}f(x)\right), \qquad x \in \mathbb{R},$$

is selfadjoint in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$, its essential spectrum coincides with $\mathbb{R}$ and at least one of the statements (i)-(iv) in Theorem 2 is true. Here it is verified numerically that the nonreal eigenvalues accumulate to zero (blue points in Figure 1). This is done as follows: Fix a solution $f_{+,\lambda}$ of

$$-f''(x) - \frac{1}{1+|x|}f(x) = \lambda f(x), \qquad x \in \mathbb{R}_+, \tag{8}$$

which belongs to $L^2(\mathbb{R}_+)$ and a solution $f_{-,\lambda}$ of

$$f''(x) + \frac{1}{1+|x|}f(x) = \lambda f(x), \qquad x \in \mathbb{R}_-, \tag{9}$$

in $L^2(\mathbb{R}_-)$. The considerations in [6] show that a point $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of A if and only if the function m, defined by

$$m(\lambda) = \frac{f'_{+,\lambda}(0)}{f_{+,\lambda}(0)} - \frac{f'_{-,\lambda}(0)}{f_{-,\lambda}(0)}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

has a zero in λ. It is well-known that equations (8) and (9) are explicitely solvable. Here we compute $f_{+,\lambda}$ and $f_{-,\lambda}$ explicitely and determine the zeros of m numerically by using the software package Mathematica (WolframResearch). A dot in the Figures 1, 2 and 3 means that for this value of λ we find solutions $f_{\pm,\lambda} \in L^2(\mathbb{R}_\pm)$ and $m(\lambda)$ vanishes within the working default precision of Mathematica.

If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then the selfadjointness of A with respect to $[\cdot,\cdot]$ implies that the complex conjugate point $\bar{\lambda}$ is also an eigenvalue. Moreover, the property $V(x) = V(-x)$ implies that besides λ also $-\lambda$ is an eigenvalue of A and the function $x \mapsto f(-x)$ is an eigenfunction at $-\lambda$ whenever $x \mapsto f(x)$ is an eigenfunction at λ. Therefore the spectrum of the indefinite differential operator A is symmetric with respect to the real and imaginary line.

Not surprisingly the potential

$$V(x) = -\frac{5}{1+|x|}, \qquad x \in \mathbb{R}, \tag{10}$$

generates complex eigenvalues of A with the same qualitative behaviour as the potential in (7), cf. Figure 2. The eigenvalues of A in the upper halfplane for the potential in (10) lie below the eigenvalues for the potential in (7).

As a third example a "combination" of the above potentials in (7) and (10) is considered which does not satisfy the symmetry condition $V(x) = V(-x)$, namely

$$V(x) = \begin{cases} -(1+x)^{-1}, & x > 0, \\ -5(1-x)^{-1}, & x < 0, \end{cases} \tag{11}$$

is considered. Here the nonreal eigenvalues of A also accumulate to zero but are not symmetric with respect to the imaginary axis, see Figure 3.

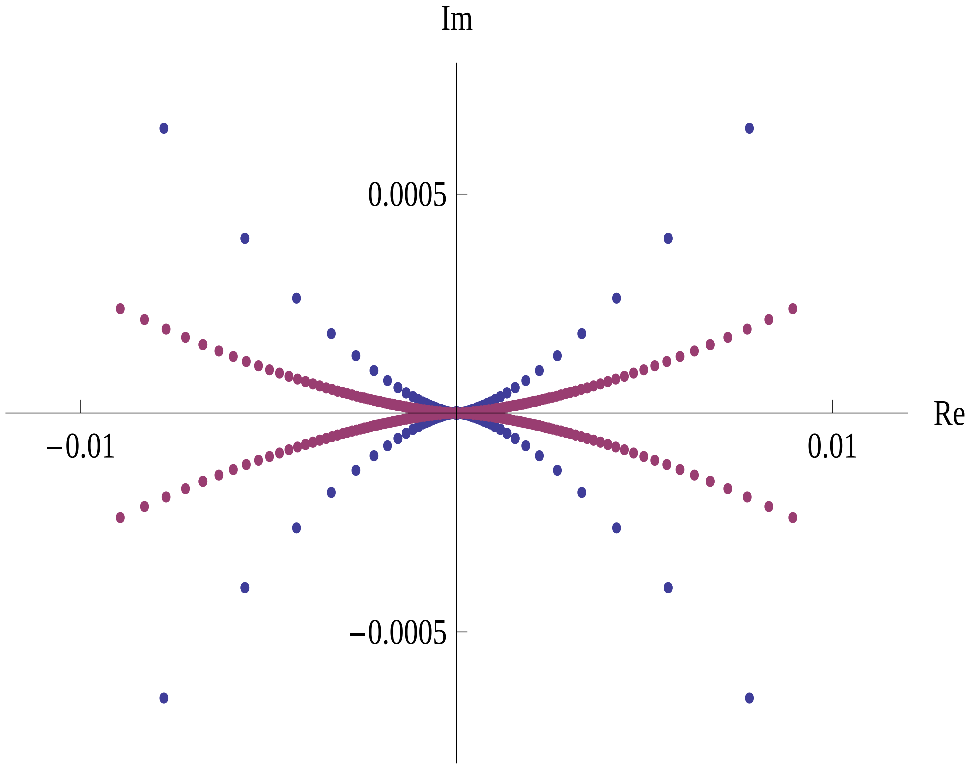

Figure 2. Accumulation of complex eigenvalues of the operator A for the potentials $V(x) = -(1+|x|)^{-1}$ (blue points) and $V(x) = -5(1+|x|)^{-1}$ (red points).

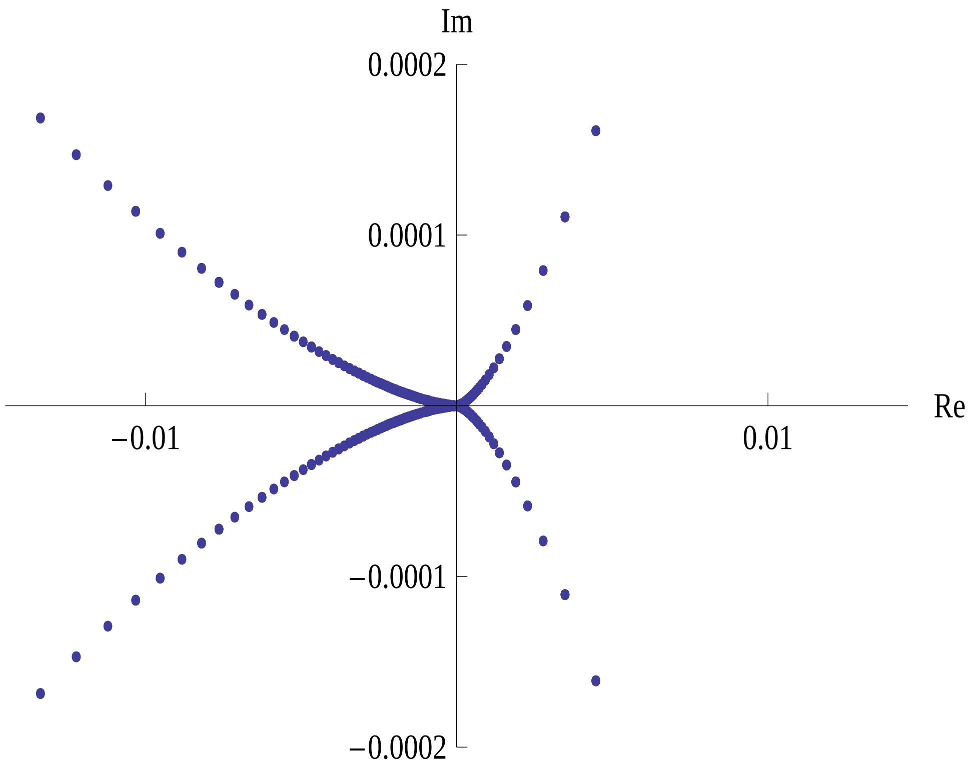

Figure 3. Accumulation of complex eigenvalues of the operator A for the nonsymmetric potential V in (11).

2. Proofs of Theorem 1 and Theorem 2

In this section a rigorous proof of Theorem 1 and Theorem 2 is given with the help of modern Krein space techniques. We first briefly recall some definitions which are essential in the following.

A point λ from the approximative point spectrum of a selfadjoint operator D in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is said to be of *positive type* (*negative type*) with respect to D, if for every sequence $(x_n) \subset \operatorname{dom} D$ with $\|x_n\| = 1$ and $\|(D-\lambda)x_n\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all points of positive (negative) type by $\sigma_{++}(D)$ (resp. $\sigma_{--}(D)$). For a detailed study and further properties of spectral points of positive and negative type we refer to [2, 13, 17, 18] and [20]. The following definition can be found in a slightly more general form in [12, 13].

Definition 1 *A selfadjoint operator D in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is said to be* definitizable over $\overline{\mathbb{C}}\backslash\{0\}$ *if the nonreal spectrum of D consists of isolated points which are poles of the resolvent of D, no point of $\overline{\mathbb{R}}\backslash\{0\}$ is an accumulation point of $\sigma(D) \cap \mathbb{C}\backslash\mathbb{R}$ and the following holds:*

- (i) *Every point $\mu \in \overline{\mathbb{R}}\backslash\{0\}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that the spectral points in each component of $I_\mu\backslash\{\mu\}$ belong either to $\sigma_{++}(D)$ or to $\sigma_{--}(D)$.*
- (ii) *For every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \overline{\mathbb{R}}\backslash\{0\}$, there exist $m \geq 1$, $M > 0$ and an open neighborhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that*

$$\|(D-\lambda)^{-1}\| \leq M(1+|\lambda|)^{2m-2}\,|\operatorname{Im}\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U}\backslash\mathbb{R}$.

We note that if the set $\overline{\mathbb{R}}\backslash\{0\}$ in the above definition is replaced by $\overline{\mathbb{R}}$, then the selfadjoint operator D is *definitizable* (over $\overline{\mathbb{C}}$) in the usual sense, i.e. the resolvent set of D is nonempty and there exists a real polynomial p such that $[p(D)x, x] \geq 0$ holds for all $x \in \operatorname{dom} p(D)$, cf. [13, Theorem 4.7] and [16].

The essence in the proof of Theorem 2 is to verify that the conditions $\lim_{x\to\pm\infty} V(x) = 0$ and (6) on the potential V imply that a certain fundamentally reducible differential operator in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$ is definitizable over $\overline{\mathbb{C}}\backslash\{0\}$ but not definitizable (over $\overline{\mathbb{C}}$). Then a recent result on finite rank perturbations of locally definitizable operators implies also non-definitizability of the indefinite Sturm-Liouville operator A in a neighborhood of zero and it follows that at least one of the statements (i)-(iv) in Theorem 2 holds. A similar type of argument also implies the assertion in Theorem 1, cf. [3, 15].

2.1. The differential operators T_+ and T_-

Let $V_+ := V \upharpoonright \mathbb{R}_+$ and $V_- := V \upharpoonright \mathbb{R}_-$ be the restrictions of the potential $V \in L^1_{\rm loc}(\mathbb{R})$ onto the positive and negative halfaxis, respectively. Since the differential expression $-\frac{d^2}{dx^2} + V$ is in the limit point case at $+\infty$ and $-\infty$, it follows that the differential expressions $-\frac{d^2}{dx^2} + V_+$ and $-\frac{d^2}{dx^2} + V_-$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, are in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Furthermore, $V \in L^1_{\rm loc}(\mathbb{R})$ implies $V_+ \in L^1[0,b)$ and $V_- \in L^1(a,0]$ for any $b \in \mathbb{R}_+$ and $a \in \mathbb{R}_-$, and hence 0 is a regular endpoint for both differential expressions on the half axes.

Consider the differential operators

$$(T_+ f_+)(x) = -f_+''(x) + (V_+ f_+)(x), \qquad x \in \mathbb{R}_+,$$

and

$$(T_- f_-)(x) = -f_-''(x) + (V_- f_-)(x), \qquad x \in \mathbb{R}_-,$$

defined on the dense subspaces

$$\operatorname{dom} T_+ = \left\{ f_+ \in L^2(\mathbb{R}_+) : f_+, f_+' \text{ a.c.}, -f_+'' + V_+ f_+ \in L^2(\mathbb{R}_+), f_+(0) = 0 \right\}$$

and

$$\operatorname{dom} T_- = \left\{ f_- \in L^2(\mathbb{R}_-) : f_-, f_-' \text{ a.c.}, -f_-'' + V_- f_- \in L^2(\mathbb{R}_-), f_-(0) = 0 \right\}$$

in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively. Here 'a.c.' is used as an abbreviation for 'absolutely continuous'. As the functions in $\operatorname{dom} T_+$ and $\operatorname{dom} T_-$ satisfy Dirichlet boundary conditions at the regular endpoint 0, it follows that T_+ and T_- are selfadjoint in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively. Since

$$\lim_{x\to+\infty} V_+(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} V_-(x) = 0$$

holds, the essential spectra of T_+ and T_- are

$$\sigma_{\rm ess}(T_+) = \sigma_{\rm ess}(T_-) = [0,\infty).$$

Furthermore, by (6) we have

$$\limsup_{x\to+\infty} x^2 V_+(x) < -\frac{1}{4} \quad \text{or} \quad \limsup_{x\to-\infty} x^2 V_-(x) < -\frac{1}{4}$$

and therefore the negative eigenvalues of T_+ or T_-, respectively, accumulate to zero (see, e.g., [11, XIII.7 Corollary 57]) and T_+ and T_- are semibounded from below, say, e.g., by the negative constants k_+ and k_-. Clearly, the operator $-T_-$ is also selfadjoint in $L^2(\mathbb{R}_-)$, its essential spectrum coincides with $(-\infty, 0]$, $-T_-$ is semibounded from above by $-k_-$ and the positive eigenvalues of $-T_-$ accumulate to zero if and only if the negative eigenvalues of T_- accumulate to zero.

2.2. Spectral properties of the operator $T_+ \times -T_-$ in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$

Let us identify the orthogonal sum of the spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ with $L^2(\mathbb{R})$. Then the diagonal operator matrix

$$B = \begin{pmatrix} T_+ & 0 \\ 0 & -T_- \end{pmatrix}, \quad \operatorname{dom} B = \operatorname{dom} T_+ \oplus \operatorname{dom} T_-, \tag{12}$$

with respect to the decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ is selfadjoint in the Hilbert space $L^2(\mathbb{R})$. Moreover, the spectrum of B is the union of the spectra of T_+ and $-T_-$, that is, the essential spectrum of B coincides with the whole real line,

$$\sigma_{\text{ess}}(B) = \mathbb{R}, \tag{13}$$

and there exists a sequence of positive (embedded) eigenvalues of B accumulating to zero or a sequence of negative (embedded) eigenvalues of B accumulating to zero. Observe that B is not a "usual" differential operator in $L^2(\mathbb{R})$ since for a function $f = f_+ \oplus f_-$ in $\operatorname{dom} B = \operatorname{dom} T_+ \oplus \operatorname{dom} T_-$ the derivative f' need not to be continuous at zero.

In the sequel B will be regarded as an operator in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$, where the indefinite inner product is given by (5). Clearly,

$$[f_+ \oplus f_-, g_+ \oplus g_-] = \int_{\mathbb{R}_+} f_+(x)\overline{g_+(x)}\,dx - \int_{\mathbb{R}_-} f_-(x)\overline{g_-(x)}\,dx$$

for $f = f_+ \oplus f_-$, $g = g_+ \oplus g_- \in L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ and if $(\cdot,\cdot)_+$ and $(\cdot,\cdot)_-$ denote the Hilbert scalar products in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, then $[f,g] = (f_+,g_+)_+ - (f_-,g_-)_-$ holds. Now the selfadjointness of T_+ and $-T_-$ implies the selfadjointness of B in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$.

Let $\lambda \in (0,\infty)$. Then λ is necessarily a spectral point of T_+ (and hence B) and there exists a sequence $(f_{n,+}) \subset \operatorname{dom} T_+$ such that $\|f_{n,+}\| = 1$ and $\|(T_+ - \lambda)f_{n,+}\| \to 0$ for $n \to \infty$. Obviously, here $\liminf_{n\to\infty}[f_{n,+} \oplus 0, f_{n,+} \oplus 0] > 0$ is true. If the point $\lambda \in (0,\infty)$ does not belong to the spectrum of $-T_-$ at the same time, then every sequence $(f_{n,+} \oplus f_{n,-}) \subset \operatorname{dom} B$ with the properties $\|f_{n,+} \oplus f_{n,-}\| = 1$ and

$$\lim_{n\to\infty} \|(B-\lambda)(f_{n,+} \oplus f_{n,-})\| = 0$$

satisfies

$$\liminf_{n\to\infty} [f_{n,+} \oplus f_{n,-}, f_{n,+} \oplus f_{n,-}] > 0.$$

Therefore, such a point λ in the spectrum of B is a spectral point of positive type of B, $\lambda \in \sigma_{++}(B)$. Hence the set $(0,\infty)\backslash\sigma(-T_-)$ consists of spectral points of positive type of B. Furthermore, each $\lambda \in (0,\infty) \cap \sigma(-T_-)$ is an eigenvalue of $-T_-$ (and an embedded eigenvalue of B) and every corresponding eigenfunction $f_- \in \operatorname{dom} T_-$ satisfies

$$[0 \oplus f_-, 0 \oplus f_-] = -(f_-, f_-)_- < 0,$$

i.e., λ is not a spectral point of positive type of B.

If $\mu \in (-\infty,0)\backslash\sigma(T_+)$, then very similar arguments as above imply that for every sequence $(g_{n,+}\oplus g_{n,-}) \subset \operatorname{dom} B$ with the properties $\|g_{n,+}\oplus g_{n,-}\| = 1$ and $\lim_{n\to\infty}\|(B-\mu)(g_{n,+}\oplus g_{n,-})\| = 0$ necessarily

$$\limsup_{n\to\infty}\,[f_{n,+}\oplus f_{n,-}, f_{n,+}\oplus f_{n,-}] < 0$$

holds, i.e., μ is a spectral point of negative type of B; $\mu \in \sigma_{--}(B)$. Any $\mu \in (-\infty,0)\cap\sigma(T_+)$ is an eigenvalue of T_+ (and an embedded eigenvalue of B) and every corresponding eigenfunction $g_+ \in \operatorname{dom} T_+$ satisfies

$$[g_+\oplus 0, g_+\oplus 0] = (g_+,g_+)_+ > 0,$$

so that μ is not a spectral point of negative type of B.

Summing up we have proved the following statements on the spectral properties of the operator B. Observe that assertion (iv) follows from the fact that the positive eigenvalues of T_+ or the negative eigenvalues of $-T_-$ accumulate to zero.

Lemma 1 *Let B be the selfadjoint operator in the Krein space $(L^2(\mathbb{R}),[\cdot,\cdot])$ from* (12) *and let T_+ and T_- be the differential operators from Section* 2.1 *with lower bounds k_+ and k_-. Then the following holds.*

(i) $\sigma(B) = \sigma_{\mathrm{ess}}(B) = \mathbb{R}$ *and* $\rho(B) = \mathbb{C}\backslash\mathbb{R}$*;*

(ii) $(0,\infty)\backslash\sigma(-T_-) = \sigma_{++}(B)$ *and* $(-k_-,\infty) \subset \sigma_{++}(B)$*;*

(iii) $(-\infty,0)\backslash\sigma(T_+) = \sigma_{--}(B)$ *and* $(-\infty,k_+) \subset \sigma_{--}(B)$*;*

(iv) *For every $\varepsilon > 0$ at least one of the following statements is true:*
   - (a) *there exist an eigenvalue $\lambda \in (0,\varepsilon)$ of B and a corresponding eigenfunction f_λ with $[f_\lambda,f_\lambda] < 0$;*
   - (b) *there exist an eigenvalue $\mu \in (-\varepsilon,0)$ of B and a corresponding eigenfunction g_μ with $[g_\mu,g_\mu] > 0$.*

Moreover, since T_+ and $-T_-$ are selfadjoint operators in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, it follows that the norm of the resolvent of the operator B can be estimated by

$$\|(B-\lambda)^{-1}\| \le \frac{1}{|\operatorname{Im}\lambda|}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Therefore the operator B is definitizable over the domain $\overline{\mathbb{C}}\backslash\{0\}$ and B is not definitizable (over $\overline{\mathbb{C}}$). The reason for non-definitizability in a neighborhood of zero is the property (iv) in Lemma 1.

2.3. The indefinite Sturm-Liouville operator A and the operator B

The intersection (in the sense of graphs) of the indefinite Sturm-Liouville operator A in (4) and the operator B is given by

$$Sf := Af = Bf, \quad f \in \operatorname{dom} S := \{y \in \operatorname{dom} A \cap \operatorname{dom} B : Ay = By\}\,. \tag{14}$$

A function $f = f_+ \oplus f_- \in L^2(\mathbb{R})$ belonging to

$$\operatorname{dom} A \cap \operatorname{dom} B = \operatorname{dom} A \cap (\operatorname{dom} T_+ \oplus \operatorname{dom} T_-)$$

is absolutely continuous and has an absolutely continuous derivative f' such that $-f'' + Vf$ belongs to $L^2(\mathbb{R})$ and $f(0) = f_+(0) = f_-(0) = 0$ holds. For such a function we have

$$Af = \operatorname{sgn}(\cdot)\,(-f'' + Vf) = \bigl(-f_+'' + V_+ f_+\bigr) \oplus \bigl(f_-'' - V_- f_-\bigr) = Bf,$$

that is, A and B coincide on the dense subspace $\operatorname{dom} A \cap \operatorname{dom} B$ of $L^2(\mathbb{R})$ and therefore the operator S in (14) is a densely defined closed operator which is a one-dimensional restriction of A and B.

With the help of the asymptotic behaviour of certain Titchmarsh-Weyl functions corresponding to the operators T_+ and T_- it can be shown that the resolvent set $\rho(A)$ of A is nonempty, see, e.g., [3, Corollary 3.4] or [15, Section 2]. Now $\dim(A/S) = \dim(B/S) = 1$ implies

$$\dim\Bigl(\operatorname{ran}\,\bigl((A-\lambda)^{-1} - (B-\lambda)^{-1}\bigr)\Bigr) = 1, \qquad \lambda \in \rho(A) \cap \rho(B), \tag{15}$$

so that A can be viewed as a one-dimensional perturbation in resolvent sense of the operator B. Thus the essential spectra of A and B coincide, $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B) = \mathbb{R}$ (see (13) and Lemma 1), and the nonreal spectrum of A consists of eigenvalues. As $-\frac{d^2}{dx^2} + V$ is in the limit point case at $\pm\infty$ the corresponding geometric eigenspaces are one-dimensional.

Furthermore, by [4, Theorem 2.2] the operator A is definitizable and non-definitizable over the same domains as B, in particular, A is definitizable over $\overline{\mathbb{C}}\backslash\{0\}$ and not definitizable (over $\overline{\mathbb{C}}$). As a consequence of the definitizability over $\overline{\mathbb{C}}\backslash\{0\}$ we find the remaining statements of Theorem 1. Non-definitizability in a neighborhood of zero can have three different reasons, firstly complex eigenvalues may accumulate to zero, secondly the growth of the resolvent of A may not be of finite order, or thirdly for each $\varepsilon > 0$ the interval $(-\varepsilon, 0)$ contains spectral points of positive and negative type of A or the interval $(0, \varepsilon)$ contains spectral points of positive and negative type of A. Since [5, Theorem 2.4] and Lemma 1 (ii) imply that $(-\infty, 0)$ and $(0, \infty)$, with the possible exception of a discrete set, belong to $\sigma_{--}(A)$ and $\sigma_{++}(A)$, respectively, the third option for non-definitizability of A in a neighborhood of zero holds if and only if there exists a sequence of positive eigenvalues of A accumulating to zero with corresponding $[\cdot,\cdot]$-negative eigenfunctions or there exists a sequence of negative eigenvalues of A accumulating to zero with corresponding $[\cdot,\cdot]$-positive eigenfunctions. This completes the proof of Theorem 2.

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